Phys 410
Fall 2013
Lecture #19 Summary
5 November, 2013

Up to this point we have considered Newtonian dynamics and Lagrangian dynamics. Now we consider Hamiltonian dynamics. The Lagrangian is written in terms of n generalized coordinates and their time derivatives. This set of parameters constitutes a $2n-$ dimensional **state space**. The Hamiltonian is written in terms of the generalized coordinates and their conjugate momenta, defined as $p_i = \partial\mathcal{L}/\partial\dot{q}_i$. This set of $2n$ parameters constitutes **phase space**.

One can solve the n canonical momentum equations for $\dot{q}_i$ in terms of the coordinates q_i and momenta p_i to arrive at $\dot{q}_i = \dot{q}_i(q_i, p_i\ )$. With this, one can express the Hamiltonian in terms of coordinates and momenta alone, essentially employing a Legendre transformation to move from $(q_i,\ \dot{q}_i)$ to $(q_i, p_i\ )$ as the independent variables. Taking the derivative of the Hamiltonian with respect to q_i and p_i, one finds Hamilton's equations: $\dot{q}_i = \partial\mathcal{H}/\partial p_i$ and $\dot{p}_i = -\partial\mathcal{H}/\partial q_i$, $i = 1, \dots, n$. This is a set of $2n$ first-order differential equations, as opposed to the set of n second-order differential equations one gets from Lagrange's equations.

The Hamiltonian dynamics formulation is useful for quantum mechanics and for classical statistical mechanics. As a way of solving classical mechanics problems it has few advantages over Lagrangian dynamics.

We then considered the Poisson Bracket (PB), which is defined as follows. Consider two dynamical functions of the generalized coordinates and conjugate momenta: $g(\vec{q},\vec{p})$ and $h(\vec{q},\vec{p})$. Examples include angular momentum, linear momentum, mechanical energy, kinetic energy, etc. Define the PB of g, h as $[g,h] \equiv \sum_{i=1}^{n}\left\{\frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}\right\}$. One can show quite easily that the following statements are true about the PB: $\frac{dg}{dt} = [g,\mathcal{H}] + \frac{\partial g}{\partial t}$, $\dot{q}_j = [q_j,\mathcal{H}]$, $\dot{p}_j = [p_j,\mathcal{H}]$, $[q_j,q_k] = 0$, $[p_j,p_k] = 0$, and most interestingly $[q_j,p_k] = \delta_{kj}$. If the PB of two dynamical quantities vanishes, then the quantities are said to commute. If the PB of two dynamical quantities is equal to 1, then the quantities are said to be canonically conjugate. Any dynamical quantity that commutes with the Hamiltonian and is not explicitly time dependent is a constant of the motion of the system. Also note from the definition of the PB that $[g,h] = -[h,g]$. Starting with this, Dirac noted that the essential new ingredient of quantum mechanics (QM) is that certain observables (u,v) give different answers depending on the order in which the observables operate on a QM system, or in other words $uv \neq vu$. To account for this, Dirac re-defined the PB for the quantum case as follows: $i\hbar[u,v]_{QM} \equiv uv - vu$. This leads to the following statements of the "fundamental quantum conditions" for the quantum position and momentum operators: $q_r q_s - q_s q_r = 0$, $p_r p_s - p_s p_r = 0$, and $q_r p_s - p_s q_r = i\hbar\delta_{rs}$. From this statement, one can derive many important results in quantum mechanics, as outlined in Dirac's book *Principles of Quantum Mechanics*.

The generalized coordinates and their conjugate momenta, defined as $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, constitute a set of $2n$ quantities that span **phase space**. The instantaneous state of the entire system is summarized as a single mathematical point in this phase space. Call this point $\vec{z} = (\vec{q}, \vec{p})$, where $\vec{q} = (q_1, \dots q_n)$ is an ordered list of the n generalized coordinates, and $\vec{p} = (p_1, \dots p_n)$ is the list of n conjugate momenta. Hamiltonian's equations describe how this point moves in phase space – in other words it describes the trajectory of the phase point. You can see this by taking the time derivative of $\vec{z}$ as $\dot{\vec{z}} = (\dot{\vec{q}}, \dot{\vec{p}})$, and noting the fact that : $\dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i} = f_i(\vec{q}, \vec{p})$ and $\dot{p}_i = -\partial\mathcal{H}/\partial q_i = g_i(\vec{q}, \vec{p})$, $i = 1, \dots, n$, where the vector functions $\vec{f}(\vec{q}, \vec{p})$, $\vec{g}(\vec{q}, \vec{p})$ summarize the derivatives of the Hamiltonian with respect to the coordinates and momenta. Thus we have the first order differential equation for the trajectory of the phase point: $\dot{\vec{z}} = \left(\vec{f}(\vec{q}, \vec{p}), \vec{g}(\vec{q}, \vec{p})\right) = \left(\vec{f}(\vec{z}), \vec{g}(\vec{z})\right)$. This is a deterministic equation for the evolution of the phase point. It shows that two trajectories that arise from two different initial conditions can never cross, because otherwise there would be two different trajectories arising from the same equation with the same instantaneous value of $\vec{z}$, contrary to the deterministic nature of the phase point evolution equation.

We considered the $2n = 2$ −dimensional phase space of a $n = 1$ one-dimensional harmonic oscillator. The trajectory of the phase point is an ellipse in the (x, p) phase plane.